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# Simplexity of the n-cube

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# Simplexity of the $n$-cube of Low Dimensions

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## Objectives

- Recreate the preexisting results for the $n = 4$ case independently.
- Find a bound for the $n = 5$ case by a method besides brute force.
- Find a general pattern in the number of possible external faces of a simplex with respect to its possible volume.
- Find a general pattern in the minimal number of simplices for some $n$-cube.

## Introduction

This project studied mathematical objects called $n$-cubes, which, like a square (2-cube) or a (3-)cube, have $\frac{n2^n}{2}$ edges such that all edges are the same length and any two edges which touch are perpendicular to each other in some plane, and their relation to mathematical objects called $n$-simplices, which, like triangles (2-simplices), have $n+1$ vertices with no more than 3 in the same plane. Specifically, it studied how few simplices are needed to cover a $n$-cube for a particular value of $n$. The simplest case, when $n = 2$, simply shows how a square (2-cube) can be covered with two triangles (2- simplices), but any higher case has several different coverings possible. This project was concerned with finding a lower bound on the minimal number of simplices needed to cover the 4-cube and 5-cube.

The project was based upon a line of papers by Mara, Heiman, Cottle, Salee, and others using various techniques to tighten the bounds on optimal solutions for coverings (or triangulations) of the $n$-cube for particular values of $n$ or for a general $n$.

## Definitions

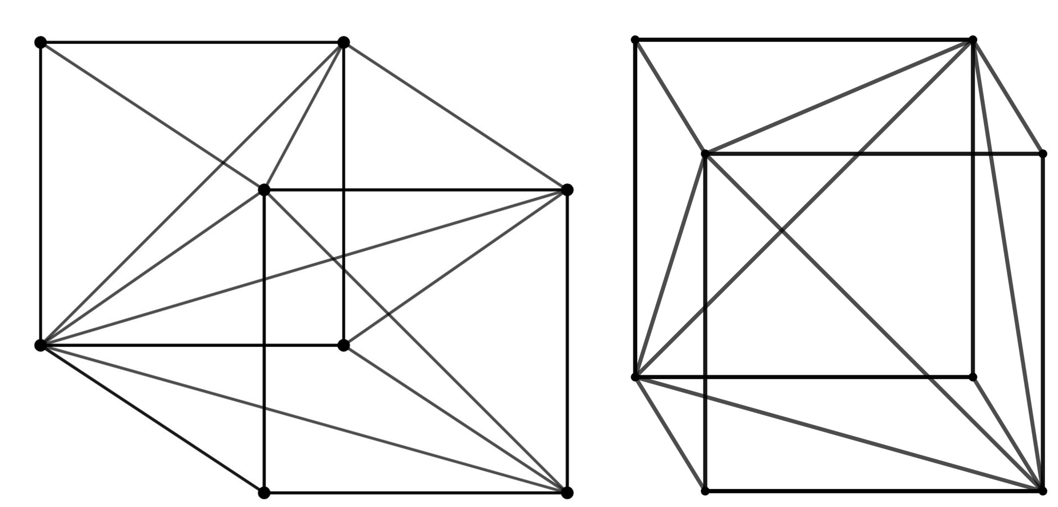
- An ***n*-simplex**, $S^n$, is an $n$-polytope with $n+1$ vertices, $V = \{v_0, v_1, \dots, v_n\}$, such that $v_i - v_j$ is linearly independent, $\forall i, j \in V$.
- A **triangulation** of an $n$-polytope, $P$, is a finite set $S$ of $n$-simplices such that $S$ covers $P$ completely and disjointly, or symbolically:
  1. $P = \cup S$
  2. For all $a, b \in S$, $a \cap b$ is a face of both $a$ and $b$ or $a \cap b = \emptyset$.

## Process

In broad strokes, the process was as follows:

- Take simple geometric object, in this case, the $n$-cube
- Find a map to an algebraic object, in this case, the matrix representation of the $n$-cube
- Associate geometric quality with algebraic quantity, in this case, the volume of the simplices and the value of the determinant of a matrix which describes it.
- Use the known quantities to set up systems of inequalities.
- Solve the systems using linear programming algorithms

## $n = 3$ case



(a) Trivial triangulation of the 3-cube (b) Optimal triangulation of the 3-cube

## Mathematical Section

In order to find the main result, a bound on the volume of a simplex was needed.

There are ways to represent a simplex as a square matrix $C$ with only 1 or 1 in its entries. This allows the use of Hadamard's inequality (which concerns the maximum value of the determinant of a certain class of square matrices) to find a bound on the determinant of $C$, namely:

$$|\det C| \leq \left|\frac{(n)^{\frac{n}{2}}}{2^{n-1}}\right| \qquad (1)$$

Since the volume of all the simplices in a triangulation of a cube is equal to the volume of the whole cube, a bound on the volume of a simplex leads to a bound on the number of possible simplices in a triangulation of a cube.

## Main Result

Let $T^n$ be a triangulation of $I^n$. Then,

(1) $\min |T^3| \geq 5$. (2) $\min |T^4| \geq 16$. (3) $\min |T^5| \geq 60$.

## Conclusion

While doing the research for this project, I found that the actual optimal value of $|T^5| = 67$. The methods used to discover this, however, involved computing every possible 5-simplex and and checking possible triangulations by computer. This project was the beginnings of an attempt to replicate this result by a more elegant method.

I found, however, that the method I used was not powerful enough to develop new independent results, and so more research would be needed.

## References

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